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EXTREMAL OVERLAP
WITH RESPECT TO TRANSLATION

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled EXTREMAL OVERLAP WITH RESPECT TO TRANSLATION submitted by MANGESH GANESH MURDESHWAR in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

By an overlap problem in number theory, we mean a problem of the following kind: Let A and B be sets of integers satisfying certain conditions. What is the largest number λ such that for every pair of sets A, B with the given conditions, the maximum overlap of A and the translates of B is $\geq \lambda$?

These problems are easily generalized to point-sets and to functions.

In this thesis, we consider overlap problems of Erdős, Czipszer, Mycielski and others. These problems are generalized to functions and certain estimates are obtained. It is further shown that some of these estimates imply similar estimates in the original number-theoretic case and its point-set analogue, and that these estimates constitute an improvement over previously known results.

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CHAPTER I

INTRODUCTION

§1.1 A Problem of Erdős

In 1955, P. Erdős [1] proposed the following problem:

Let the set of integers $1,2,\ldots,2n$ be separated into two (disjoint) classes

$$A: \quad a_1 < a_2 < \ldots < a_n$$

$$B: \quad b_1 < b_2 < \ldots < b_n \quad .$$

Let

$$M_k = \sum_{a_i - b_j = k} 1$$

$$M = M(n) = \min_{A,B} \max_k M_k$$

where the minimum is taken over all separations A and B. Find or estimate M.

We shall refer to this problem as the overlap problem of Erdős.

Erdös [2] had conjectured that $M = \frac{n}{2}$, for even n. (Erdös had stated the problem for even n; but this restriction has been removed in this presentation.) ~~But~~ Later, using a probability argument he proved that for large n,

$$M \leq \frac{4}{9} n \quad .$$

T. S. Motzkin, K. E. Ralston and J. L. Selfridge [6] used the electronic computer SWAC to find that for $n = 15$ (and multiples of 15)

$$M \leq (0.4)n \quad .$$

As for the bounds on the other side, Erdös [2] showed that

$$M \geq (.25)n \quad . \tag{1.1}$$

This estimate was improved upon by P. Scherk (written communication) who proved that

$$M > (1 - \frac{1}{\sqrt{2}})n \doteq 0.2929\ n \quad . \tag{1.2}$$

A further improvement was obtained in 1958 by L. Moser [4] who showed that

$$M > \frac{\sqrt{2}}{4}\ (n-1) \doteq 0.3535(n-1) \quad . \tag{1.3}$$

He further mentions that by combining his method with that of P. Scherk, one can get

$$M > \sqrt{4-\sqrt{15}}\,(n-1) \doteq 0.3563(n-1) \quad . \tag{1.4}$$

This last estimate remains the best known so far.

Note that in this overlap problem of Erdős, (i) the sets A and B have the same number of elements, (ii) A and B are complementary.

By omitting either or both these conditions, one can obtain a generalization. In the following problem, both conditions have been dropped, whereas in the problem N2, condition (i) has been omitted but condition (ii) has been retained.

<u>Problem N1</u>.

Let

$$A: \quad a_1 < a_2 < \ldots < a_k$$

$$B: \quad b_1 < b_2 < \ldots < b_\ell$$

be subsets of the set of integers $1,2,\ldots,n$;

$$M_r = \sum_{a_i - b_j = r} 1 \qquad -n < r < n \tag{1.5}$$

$$M = \max_r M_r \tag{1.6}$$

$$\lambda_{N1} = \min_{A,B} M \tag{1.7}$$

where the minimum is taken over all subsets A and B.

To find or estimate λ_{N1}.

In Chapter II, we obtain lower bounds for λ_{N1}.

Problem N2.

Let A be the set defined in N1, and B its complement relative to the set $\{1,2,...,n\}$, so that $\ell = n-k$. Let M_r and M be defined by (1.5) and (1.6) respectively, and

$$\lambda_{N2} = \min_{A} M \tag{1.8}$$

the minimum being taken over all subsets A of $\{1,2,...,n\}$ with k elements.

S. Świerczkowski [9] proved that

$$\lambda_{N2} \geq \frac{n}{5} \left(2 - \sqrt{4 - 10k(n-k)/n^2}\right) . \tag{1.9}$$

The overlap problem of Erdős as well as problems N1 and N2 can be generalized in another direction:

(i) to point-sets,

(ii) to real functions of a real variable,

(iii) to real functions of several variables.

We shall now state the generalizations of N1 and N2 to point-sets and call them P1 and P2 respectively.

Problem P1.

Let X, Y be Lebesgue measurable subsets of the interval $[0,1]$ with

$$(1.10)\qquad \begin{cases} \alpha = m(X) \\ \beta = m(Y) . \end{cases}$$

If, for a real number t, Y_t denotes the translated set

$$\{y + t \mid y \in Y\} ,$$

let

$$(1.11)\qquad M_t = m(X \cap Y_t)$$

$$(1.12)\qquad M = \sup_t M_t$$

$$(1.13)\qquad \lambda_{P1} = \inf_{X,Y} M$$

where the infimum is taken over all subsets X and Y of $[0,1]$, subject to conditions (1.10). α, β are thought of as fixed though arbitrary.

Jan Mycielski, who was the first one to consider this problem, proved that

$$\lambda_{P1} \geq 1 - \sqrt{1 - \alpha\beta} \tag{1.14}$$

This result without proof has been mentioned by Swierczkowski [9]. Both the result and the proof were recently communicated to us by Mycielski.

Mycielski's method employs an infinite process involving continued fractions. In Chapter II we shall prove (1.14) more simply, by generalizing the problem to functions.

An improvement over (1.14) is also obtained in Cahpter II.

Problem P2.

Let X, Y be defined as in P1 with the further condition that

$$Y = [0,1] - X$$

so that $\alpha + \beta = 1$.

Let M_t and M be given by (1.11) and (1.12) respectively,

and

$$\lambda_{P2} = \inf_X M \tag{1.15}$$

where the infimum is taken over all subsets X of Lebesgue measure α. (α is fixed, though arbitrary.)

This problem was first considered by Swierczkowski [9] who showed that, as a consequence of (1.9),

$$\lambda_{P2} \geq \frac{1}{5}\left(2 - \sqrt{4 - 10\alpha(1-\alpha)}\right) \quad . \tag{1.16}$$

A special case of P2, namely $\alpha = \frac{1}{2}$, was dealt with by H. Schell [8] who proved

$$\lambda_{P2} \geq \frac{2 - \sqrt{2}}{4} \quad . \tag{1.17}$$

In Chapter III, we shall obtain an improvement over (1.16).

Generalizations of P1 and P2 to functions of one variable will be called F1 and F2 respectively. These are stated at the beginning of Chapter II and Chapter III respectively, in which they are dealt with in detail. Two-variable versions of both these problems are considered in Chapter IV.

§1.2 A Problem of Czipszer

The following overlap problem is due to J. Czipszer (written

communication from P. Erdös):

Let

$$A: \quad a_1, a_2, a_3, \ldots, a_n$$

be a set of positive integers, and for each integer k, let

$$A_k = \{a_i + k \mid a_i \in A\} \quad .$$

Define

$$M_k = |A_k - A|$$

where $|\cdot|$ denotes the number of elements; and

$$M = \min_{A} \max_{-n \leq k \leq n} M_k \quad .$$

Find or estimate M.

Czipszer proved that

$$M \geq \frac{n}{2} \quad . \tag{1.18}$$

The problem of Czipszer can also be generalized to point-sets and to functions. The former will be stated now, and the latter will be dealt with in detail in Cahpter V.

Point-set Analogue of Czipszer problem.

Let X be a Lebesgue measurable set of real numbers with $m(X) = 1$. For each real t, let

$$X_t = \{x + t \mid x \in X\} .$$

Define

$$M_t = m(X_t - X) \tag{1.19}$$

$$M = \sup_{|t| \leq 1} M_t \tag{1.20}$$

$$\mu_p = \inf_X M \quad . \tag{1.21}$$

Erdös, using probabilistic argument, proved that

$$\mu_p \geq \frac{1}{2} \tag{1.22}$$

which corresponds to (1.18).

Świerczkowski [10] proved that

$$\frac{1}{2} + \frac{1}{18 + 8\sqrt{5}} \leq \mu_p \leq \frac{2}{3} \tag{1.23}$$

i.e.,

$$0{\cdot}5278 \leq \mu_p \leq 0{\cdot}666\dot{6} \quad . \tag{1.24}$$

In Chapter V, the lower bound will be improved to $0{\cdot}5892$.

The layout of this dissertation is as follows:

In Chapter II we shall deal entirely with problem F1 and in Chapter III with problem F2. Two-variable versions of both F1 and F2 will be considered in Chapter IV. Chapter V will deal with the function-analogue of the problem of Czipszer and its generalization to k variables. In the Appendix, we shall discuss the relations between the constants involved in the different cases of the same problem (e.g. λ_{F1}, λ_{P1} and λ_{N1}), and show in effect that most of our results derived for the function-case imply similar results for the other two cases.

CHAPTER II

PROBLEM F1

§2.1

Let $\mathcal{F}$ denote the class of all functions* f satisfying the following conditions:

(2.1) $$f \in L[0,1]$$

(2.2) $$\begin{cases} 0 \leq f(x) \leq 1 & \text{for } x \in [0,1] \\ f(x) = 0 & \text{for } x \notin [0,1] \end{cases}$$

(2.3) $$\int_0^1 f(x)dx = \alpha \quad ,$$

and $\mathcal{G}$ denote the class of all functions g such that

(2.4) $$g \in L[0,1]$$

(2.5) $$\begin{cases} 0 \leq g(x) \leq 1 & \text{for } x \in [0,1] \\ g(x) = 0 & \text{for } x \notin [0,1] \end{cases}$$

(2.6) $$\int_0^1 g(x)dx = \beta \quad .$$

α and β are thought of as fixed, but arbitrary.

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* When no ambiguity results, we will use the word "function" to signify real-valued function of one or more real variables.

Define

$$(2.7) \qquad \mathcal{M}(t) = \mathcal{M}(f,g;t) = \int_0^1 f(x)g(x+t)dx$$

$$(2.8) \qquad M = M(f,g) = \sup_t \mathcal{M}(t)$$

$$(2.9) \qquad \lambda_{F1} = \inf M$$

where the infimum is taken over all $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

The problem is to find or estimate λ_{F1}.

In this chapter, we shall obtain upper and lower estimates for λ_{F1}. We need to prove some lemmas for this purpose.

LEMMA 2.1

$\mathcal{M}(t)$ is continuous and hence integrable on $(-\infty,\infty)$.

Proof: $$\mathcal{M}(t+h) - \mathcal{M}(t) = \int_0^1 f(x)[g(x+t+h)-g(x+t)]dx .$$

Therefore,

$$|\mathcal{M}(t+h) - \mathcal{M}(t)| \leq \int_0^1 |f(x)[g(x+t+h)-g(x+t)]|dx$$

$$\leq \int_0^1 |g(x+t+h)-g(x+t)|dx$$

since $|f(x)| \leq 1$.

The last integral tends to zero as $h \to 0$, by the Mean-Continuity Property.

LEMMA 2.2

(2.10) $$\int_{-1}^{1} \mathcal{M}(t)dt = \alpha \beta \quad .$$

Proof:

$$\int_{-1}^{1} \mathcal{M}(t)dt = \int_{-1}^{1} dt \int_{0}^{1} f(x)g(x+t)dx$$

$$= \int_{0}^{1} f(x)dx \int_{-1}^{1} g(x+t)dt$$

$$= \int_{0}^{1} f(x)dx \int_{-1+x}^{1+x} g(u)du \qquad (u = x+t)$$

$$= \int_{0}^{1} f(x)dx \int_{0}^{1} g(u)du$$

$$= \alpha \beta \quad .$$

The change in the order of integration is justified by Fubini's Theorem.

LEMMA 2.3

For $|t| \leq 1$,

(2.11) $$\mathcal{M}(t) \leq 1 - |t| .$$

Proof: For $0 \leq t \leq 1$,

$$\mathcal{M}(t) = \int_0^1 f(x)g(x+t)dx$$

$$= \int_0^{1-t} f(x)g(x+t)dx$$

since $g(x+t) = 0$ for $x > 1 - t$. Further, using $0 \leq f(x) \leq 1$, $0 \leq g(x) \leq 1$, we get

$$\mathcal{M}(t) \leq 1 - t \quad .$$

Similarly, when $-1 \leq t < 0$,

$$\mathcal{M}(t) = \int_0^1 f(x)g(x+t)dx$$

$$= \int_{-t}^1 f(x)g(x+t)dx$$

since $g(x+t) = 0$ for $x < -t$. Once again, using the bounds on f and g,

$$\mathcal{M}(t) \leq 1 + t$$

$$= 1 - |t| \quad .$$

From Lemma 2.2, it follows immediately that

$$M \geq \frac{\alpha\beta}{2}$$

for all $f \in \mathcal{F}$, $g \in \mathcal{G}$; i.e.,

(2.12) $$\lambda_{F1} \geq \frac{\alpha\beta}{2} \quad .$$

This estimate corresponds to (1.1).

The next theorem gives an estimate which corresponds to that of Mycielski, viz. (1.14).

<u>THEOREM 1.1</u>

(2.13) $$\lambda_{F1} \geq 1 - \sqrt{1 - \alpha\beta} \quad .$$

<u>Proof</u>:

$$\alpha\beta = \int_{-1}^{1} \mathcal{M}(t)dt$$

$$= \int_{-1}^{-1+M} \mathcal{M}(t)dt + \int_{-1+M}^{1-M} \mathcal{M}(t)dt + \int_{1-M}^{1} \mathcal{M}(t)dt$$

$$\leq \int_{-1}^{-1+M} (1+t)dt + M \int_{-1+M}^{1-M} dt + \int_{1-M}^{1} (1-t)dt$$

using Lemma 2.3 and the fact $\mathcal{M}(t) \leq M$. On simplification, this yields

$$\alpha\,\beta \leq 2M - M^2$$

i.e.,

$$M^2 - 2M + \alpha\,\beta \leq 0 \quad .$$

Solving this inequality, one obtains

(2.14) $$M \geq 1 - \sqrt{1 - \alpha\beta} \quad .$$

Since the inequality (2.14) is true for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$, (2.13) follows.

We shall now obtain an upper bound for λ_{F1} .

THEOREM 2.2

There exist a function $f \in \mathcal{F}$ and a function $g \in \mathcal{G}$ for which

$$M \le \frac{\alpha\,\beta}{2 - (\alpha \wedge \beta)}$$

where $\alpha \wedge \beta$ denotes $\min(\alpha,\beta)$.

In other words,

$$\lambda_{F1} \le \frac{\alpha\,\beta}{2 - (\alpha \wedge \beta)} \quad . \tag{2.15}$$

<u>Proof</u>: Suppose, without loss of generality, that $\alpha = \alpha \wedge \beta$. Define

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{\alpha}{2}] \cup [1 - \frac{\alpha}{2}, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} \dfrac{\beta}{1 - \frac{\alpha}{2}} & \text{if } x \in (\frac{\alpha}{4}, 1 - \frac{\alpha}{4}) \\ 0 & \text{otherwise} \quad . \end{cases}$$

For these functions,

$$M = \mathcal{M}(0) = \frac{\alpha}{2} \cdot \frac{\beta}{1 - \frac{\alpha}{2}} \quad .$$

<u>Remark</u>: This theorem shows that the scope for improvement over (2.13) is very slight if any. In particular, we note that it is not possible to improve the leading term.

THEOREM 2.3

(2.16) $$\lambda_{F1} \geq \alpha + \beta - 1 \ .$$

Proof: From

$$0 \leq f(x) \leq 1$$

$$0 \leq g(y) \leq 1$$

we have

(2.17) $$(1-f(x))(1-g(y)) \geq 0 \ .$$

Hence

$$f(x)g(y) \geq f(x) + g(y) - 1 \ .$$

Then for all $f \in \mathcal{F}$, $g \in \mathcal{G}$,

$$M \geq \mathcal{M}(0) = \int_0^1 f(x)g(x)dx$$

$$\geq \int_0^1 [f(x)+g(x) - 1] \, dx$$

$$= \alpha + \beta - 1 \ .$$

Therefore,

$$\lambda_{F1} \geq \alpha + \beta - 1 \quad .$$

<u>Remark</u>: The estimate (2.15) is useful only when α and β are both "large" i.e., sufficiently close to 1.

Before proceeding to Theorem 2.4, which will be the main theorem of this chapter, we need to introduce an "auxiliary" function $\mathcal{N}(t)$ and to prove a few lemmas.

We define

$$\mathcal{N}(t) = \mathcal{M}(t) + \mathcal{M}(-t) \ , \qquad t \geq 0 \tag{2.18}$$

and make note of the following observations, some of which are obvious and others follow directly from similar results proved for $\mathcal{M}(t)$ in the preceding lemmas.

$$0 \leq \mathcal{N}(t) \leq 2 \qquad t \in [0,1], \tag{2.19}$$

$$\mathcal{N}(t) = 0 \qquad t > 1 \ , \tag{2.20}$$

(2.21) $\mathcal{N}(t)$ is continuous,

$$\int_0^1 \mathcal{N}(t)dt = \alpha\,\beta \ , \tag{2.22}$$

and

$$\mathcal{N}(t) \leq 2 - 2t \qquad 0 \leq t \leq 1 \quad . \tag{2.23}$$

Further, if

$$N = \max_t \mathcal{N}(t) \tag{2.24}$$

then

$$N \le 2M \ . \tag{2.25}$$

<u>LEMMA 2.4</u>

If E is any measurable subset of $[0,1]$, then

$$\int_E (f+g) \le \frac{N}{2} + m(E) \tag{2.26}$$

where $m(E)$ denotes the measure of E.

<u>Proof</u>: From (2.17),

$$0 \ge f(x)g(x) \ge f(x)+g(x) - 1 \ .$$

Hence

$$N \ge \mathcal{N}(0)$$

$$= 2 \int_0^1 f(x)g(x)dx$$

$$\geq \; 2 \int_E f(x)g(x)dx$$

$$\geq \; 2 \int_E \{f(x)+g(x) -1\}dx$$

$$= \; 2 \int_E \{f(x)+g(x)\}dx - 2m(E) \; .$$

This yields (2.26) .

<u>LEMMA 2.5</u>

For $\frac{1}{2} \leq t \leq 1$,

(2.27) $$\mathcal{N}(t) \leq \frac{N}{4} + 1 - t \; .$$

<u>Proof</u>:

$$\mathcal{N}(t) = \mathcal{M}(t) + \mathcal{M}(-t)$$

$$= \int_0^{1-t} f(x)g(x+t)dx + \int_t^1 f(x)g(x-t)dx$$

$$\leq \int_0^{1-t} f(x)dx + \int_t^1 f(x)dx$$

since $0 \leq g(x \pm t) \leq 1$.

Take $E = (0,1-t) \cup (t,1)$. Then,

$$(2.28) \qquad \mathcal{N}(t) \leq \int_E f(x)\,dx \quad .$$

Similarly,

$$\mathcal{N}(t) = \int_t^1 f(x-t)g(x)dx + \int_0^{1-t} f(x+t)g(x)dx$$

$$\leq \int_t^1 g(x)dx + \int_0^{1-t} g(x)dx \ ,$$

i.e.,

$$(2.29) \qquad \mathcal{N}(t) \leq \int_E g(x)dx \quad .$$

From (2.28) and (2.29),

$$\mathcal{N}(t) \leq \frac{\int_E (f(x)+g(x))dx}{2}$$

$$\leq \frac{N}{4} + \frac{m(E)}{2}$$

using Lemma 2.4. Since $m(E) = 2(1-t)$,

$$\mathcal{N}(t) \leq \frac{N}{4} + 1 - t \quad .$$

LEMMA 2.6

For $0 \leq t \leq \frac{1}{2}$,

$$\mathcal{N}(t) \leq \frac{N}{4} + 2 - 3t \quad . \tag{2.30}$$

Proof:

$$\mathcal{N}(t) = \int_0^{1-t} f(x)g(x+t)dx + \int_t^1 f(x)g(x-t)dx$$

$$= \int_0^{t} f(x)g(x+t)dx + \int_t^{1-t} f(x)g(x+t)dx$$

$$+ \int_t^{1-t} f(x)g(x-t)dx + \int_{1-t}^{1} f(x)g(x-t)dx$$

$$\leq \int_0^{t} f(x)dx + 2\int_t^{1-t} f(x)dx + \int_{1-t}^{1} f(x)dx$$

$$\leq \int_E f(x)dx + 2(1-2t)$$

where $E = (0,t)\cup(1-t,1)$.

Similarly,

$$\mathcal{N}(t) = \int_t^1 f(x-t)g(x)dx + \int_0^{1-t} f(x+t)g(x)dx$$

$$\le \int_t^1 g(x)dx + \int_0^{1-t} g(x)dx$$

$$= \int_E g(x)dx + 2\int_t^{1-t} g(x)dx$$

$$\le \int_E g(x)dx + 2(1-2t) \quad .$$

Hence,

$$\mathcal{N}(t) \le \frac{\int_E (f+g)}{2} + 2(1-2t)$$

$$\le \frac{N}{4} + \frac{m(E)}{2} + 2(1-2t)$$

by Lemma 2.4. The required result now follows on writing $m(E) = 2t$.

THEOREM 2.4

(i) If $\lambda_{F1} < \frac{1}{3}$, then

(2.31) $$\lambda_{F1} \ge \frac{1 - \sqrt{1-c\alpha\beta}}{c} \quad \text{where} \quad c = \frac{5}{4} .$$

(ii) If $\frac{1}{3} < \lambda_{F1} < \frac{1}{2}$, then

(2.32) $$\lambda_{F1} \ge \max\left\{\frac{1}{3} , \frac{1 - \sqrt{1-d\alpha\beta}}{d}\right\}$$

where $d = \frac{7}{6}$.

(iii) If $\lambda_{F1} > \frac{1}{2}$, then

(2.33) $$\lambda_{F1} \geq \max\left\{\frac{1}{2}, \alpha + \beta - 1\right\} .$$

Proof:

By definition of λ_{F1}, for each $\epsilon > 0$ there exist functions $f_\epsilon \in \mathcal{F}$ and $g_\epsilon \in \mathcal{G}$ such that

(2.34) $$M(f_\epsilon, g_\epsilon) < \lambda_{F1} + \epsilon .$$

Throughout this proof we shall write M to stand for $M(f_\epsilon, g_\epsilon)$ and N to stand for $N(f_\epsilon, g_\epsilon)$.

(i) We choose ϵ so that

$$\lambda_{F1} + \epsilon \leq \frac{1}{3} .$$

Hence

$$M < \frac{1}{3}$$

and

$$N < \frac{2}{3} .$$

Let

$$A_1 = (0,\ 1 - \frac{3N}{4})$$

$$A_2 = (1 - \frac{3N}{4}\ ,\ 1 - \frac{N}{4})$$

$$A_3 = (1 - \frac{N}{4}\ ,\ 1)\ .$$

Note that if $t \in A_2$, then

$$\begin{aligned} t &\geq 1 - \frac{3N}{4} \\ &\geq \frac{1}{2} \end{aligned}$$

since $N \leq \frac{2}{3}$; and hence Lemma 2.5 applies. Therefore

$$\begin{aligned} \alpha\beta &= \int_0^1 \mathcal{N}(t)dt \\ &= \int_{A_1} \mathcal{N}(t)dt + \int_{A_2} \mathcal{N}(t)dt + \int_{A_3} \mathcal{N}(t)dt \\ &\leq N\int_{A_1} dt + \int_{A_2} (\frac{N}{4}+1-t)dt + \int_{A_3} (2-2t)dt \end{aligned}$$

using Lemma 2.5 and the estimate (2.23).

On simplifying,

$$\alpha\beta \leq N - \frac{5}{16} N^2$$

i.e.,

$$\frac{5}{16} N^2 - N + \alpha\beta \leq 0 ,$$

which implies

$$N \geq \frac{1 - \sqrt{1 - \frac{5}{4}\alpha\beta}}{\frac{5}{8}} .$$

Since $M \geq \frac{1}{2} N$, this yields

$$M \geq \frac{1 - \sqrt{1-c\alpha\beta}}{c} \qquad \text{with} \quad c = \frac{5}{4}$$

and hence by (2.34)

$$\lambda_{F1} > \frac{1 - \sqrt{1-c\alpha\beta}}{c} - \epsilon .$$

Since ϵ is arbitrary, (2.31) follows.

(ii) In this case we choose ϵ so that

(2.35) $$\lambda_{F1} + \epsilon \leq \frac{1}{2} .$$

Therefore,

$$N < 1 .$$

Let now

$$A_1 = (0, \tfrac{2}{3} - \tfrac{N}{4})$$

$$A_2 = (\tfrac{2}{3} - \tfrac{N}{4}, \tfrac{1}{2})$$

$$A_3 = (\tfrac{1}{2}, 1 - \tfrac{N}{4})$$

$$A_4 = (1 - \tfrac{N}{4}, 1) \quad .$$

Then,

$$\alpha\beta = \int_0^1 \mathcal{N}(t)dt$$

$$= \int_{A_1} + \int_{A_2} + \int_{A_3} + \int_{A_4} \mathcal{N}(t)dt$$

$$\leq N\int_{A_1} dt_1 + \int_{A_2} (\tfrac{N}{4} + 2-3t)dt + \int_{A_3} (\tfrac{N}{4} + 1-t)dt + \int_{A_4} (2-2t)dt$$

using Lemmas 2.5 and 2.6.

Therefore

$$\alpha\beta \leq N(\tfrac{2}{3} - \tfrac{N}{4}) + (\tfrac{N}{4} + 2)(\tfrac{N}{4} - \tfrac{1}{6}) - \tfrac{3}{2}\left\{\tfrac{1}{4} - (\tfrac{2}{3} - \tfrac{N}{4})^2\right\}$$

$$+ (\tfrac{N}{4} + 1)(\tfrac{1}{2} - \tfrac{N}{4}) - \tfrac{1}{2}\left\{(1 - \tfrac{N}{4})^2 - \tfrac{1}{4}\right\}$$

$$+ \tfrac{N}{2} - \left\{1 - (1 - \tfrac{N}{4})^2\right\}$$

$$= N - \tfrac{5}{16} N^2 + \tfrac{1}{3}(\tfrac{3N}{4} - \tfrac{1}{2})^2 \quad .$$

Since $N < 1$,

$$\frac{3N}{4} - \frac{1}{2} < \frac{N}{4} .$$

Hence

$$\alpha\beta < N - \frac{5}{16} N^2 + \frac{1}{3} \left(\frac{N}{4}\right)^2$$

$$= N - \frac{7}{24} N^2$$

i.e.

$$\frac{7}{24} N^2 - N + \alpha\beta < 0 .$$

Therefore,

$$N > \frac{1 - \sqrt{1-d\alpha\beta}}{\frac{1}{2} d} \qquad \text{with} \quad d = \frac{7}{6} .$$

Hence

$$M > \frac{1 - \sqrt{1-d\alpha\beta}}{d} .$$

By (2.35),

$$\lambda_{F1} > \frac{1 - \sqrt{1-d\alpha\beta}}{d} - \epsilon .$$

Since ϵ can be arbitrarily small, (2.32) follows.

(iii) follows from hypothesis and Theorem 2.3.

By virtue of (6.1) and (6.4) of the Appendix, most of the results derived in this chapter imply similar results for λ_{P1} and λ_{N1}. In particular (2.13) implies Mycielski estimate (1.14) and also the following estimate for λ_{N1} :

(2.36) $$\lambda_{N1} \geq n \{1 - \sqrt{1 - k\ell/n^2}\} \quad .$$

Similarly from Theorem 2.3, we deduce

(2.37) $$\lambda_{P1} \geq \alpha + \beta - 1$$

(2.38) $$\lambda_{N1} \geq k + \ell - n \quad .$$

Finally, we remark that in Theorem 2.4, we may substitute λ_{P1} for λ_{F1} ; and further $\frac{\lambda_{N1}}{n}$, $\frac{k}{n}$, $\frac{\ell}{n}$ may replace λ_{P1}, α,β respectively.

§2.2 Transform Method

The result contained in Theorem 2.5 at the end of this section constitutes an improvement over Theorem 2.4 for sufficiently small values of α and β. The method uses Fourier transforms and hence may be called the Transform Method.

In this section, we consider f, g to be functions on $[-\frac{1}{2}, \frac{1}{2}]$ rather than on $[0,1]$. Clearly, this makes no difference

to our problem. The other conditions are modified accordingly, viz.:

(2.39) $$f \in L[-\tfrac{1}{2}, \tfrac{1}{2}]$$

(2.40) $$\begin{cases} 0 \leq f(x) \leq 1, & x \in [-\tfrac{1}{2}, \tfrac{1}{2}] \\ f(x) = 0, & x \notin [-\tfrac{1}{2}, \tfrac{1}{2}] \end{cases}$$

(2.41) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = \alpha$$

and

(2.42) $$g \in L[-\tfrac{1}{2}, \tfrac{1}{2}]$$

(2.43) $$\begin{cases} 0 \leq g(x) \leq 1, & x \in [-\tfrac{1}{2}, \tfrac{1}{2}] \\ g(x) = 0 & x \notin [-\tfrac{1}{2}, \tfrac{1}{2}] \end{cases}$$

(2.44) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} g(x)dx = \beta \quad .$$

Thus, (for the purposes of this section only) $\mathcal{F}$ denotes the class of all functions f satisfying (2.39) to (2.41) and $\mathcal{G}$ denotes the class of all functions g satisfying (2.42) to (2.44).

Let $g^* \in \mathcal{G}$ be defined by

$$g^*(x) = g(-x)$$

and define

$$\mathcal{M}^*(t) = \mathcal{M}^*(f,g;t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)g(t-x)dx$$

$$M^* = M^*(f,g) = \sup_t \mathcal{M}^*(t) \quad .$$

Note that

$$\mathcal{M}^*(f,g;t) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)g^*(x-t)dx$$

$$= \mathcal{M}(f,g^*;-t) \quad .$$

Taking the supremum over all t,

(2.45) $$M^*(f,g) = M(f,g^*) \quad .$$

Further, as g runs through $\mathcal{G}$, g^* also runs through $\mathcal{G}$. Therefore

(2.46) $$\lambda_{F1} = \inf_{\substack{f\in\mathcal{F}\\ g\in\mathcal{G}}} M^*(f,g) \quad .$$

We now define the following transforms:

$$F(s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{is(x+\frac{1}{2})} f(x)dx$$

$$G(s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{is(x+\frac{1}{2})} g(x)dx$$

$$H(s) = \int_{-1}^{1} e^{is(t + 1)} \mathcal{M}^*(t)dt \quad .$$

LEMMA 2.7

$$H(s) = F(s)\, G(s) \; .$$

Proof:

$$H(s) = \int_{-1}^{1} e^{is(t+1)} \mathcal{M}^*(t)dt$$

$$= \int_{-1}^{1} e^{is(t+1)} dt \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)g(t-x)dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx \int_{-1}^{1} e^{is(t+1)} g(t-x)dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx \int_{-1-x}^{1-x} e^{is(u+x+1)} g(u)du$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{is(x + \frac{1}{2})} f(x)dx \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{is(u + \frac{1}{2})} g(u)du$$

$$= F(s)\, G(s) \quad .$$

LEMMA 2.8

If $\tilde{\mathcal{M}}(t) = \mathcal{M}^*(t) - \frac{\alpha\beta}{2}$, then

$$|H(\pi)| \le \int_{-1}^{1} |\tilde{\mathcal{M}}(t)|dt \quad . \tag{2.47}$$

Proof:

$$H(\pi) = \int_{-1}^{1} e^{i\pi(t+1)}\mathcal{M}^*(t)dt$$

$$= -\int_{-1}^{1} e^{i\pi t} [\tilde{\mathcal{M}}(t) + \frac{\alpha\beta}{2}]dt$$

$$= -\int_{-1}^{1} e^{i\pi t}\tilde{\mathcal{M}}(t)dt - \frac{\alpha\beta}{2} \int_{-1}^{1} e^{i\pi t}\, dt$$

$$= -\int_{-1}^{1} e^{i\pi t}\tilde{\mathcal{M}}(t)dt \quad .$$

Hence,

$$|H(\pi)| \le \int_{-1}^{1} |\tilde{\mathcal{M}}(t)|dt \quad .$$

LEMMA 2.9

If φ is an integrable function on $[-1,1]$, and

$$\int_{-1}^{1} \varphi(x)dx = 0 \tag{2.48}$$

$$\int_{-1}^{1} |\varphi(x)|dx \geq 2K \tag{2.49}$$

$$\varphi(x) \geq -k \ , \tag{2.50}$$

then for some $x_1 \in [-1,1]$

$$\varphi(x_1) \geq \frac{K}{2 - K/k} \ . \tag{2.51}$$

<u>Proof</u>:

Let

$$E = \{x \in [-1,1] | \varphi(x) > 0\}$$

$$F = \{x \in [-1,1] | \varphi(x) \leq 0\} \ .$$

Then, by (2.48)

$$\int_E \varphi(x)dx = -\int_F \varphi(x)dx \tag{2.52}$$

and,

$$2K \leq \int_E |\varphi(x)|dx + \int_F |\varphi(x)|dx$$

$$= \int_E \varphi(x)dx - \int_F \varphi(x)dx$$

$$= 2\int_E \varphi(x)dx$$

i.e.,

$$\int_E \varphi(x) \geq K \quad . \tag{2.53}$$

Hence there exists $x_1 \in E$ such that

$$\varphi(x_1) \geq \frac{K}{m(E)}$$

$$= \frac{K}{2-m(F)} \quad .$$

From (2.52) and (2.53), we also have

$$K \leq -\int_F \varphi(x)dx$$

$$\leq k.m(F)$$

using (2.50).

Hence

$$m(F) \geq \frac{K}{k}$$

and

$$\varphi(x_1) \geq \frac{K}{2 - K/k} \quad .$$

Theorem 2.5

(2.54) $$\lambda_{F1} \geq \min\left\{ \frac{\alpha+\beta}{2} - 2\tau, \ \frac{\alpha\beta}{2}\left[1 + \frac{K}{\alpha\beta - K}\right]\right\}$$

where

(2.55) $$\pi^2 K = 2\left(1 - \cos\frac{\pi\alpha}{2}\right)\left\{\cos\pi\tau - \cos\pi\left(\tau + \frac{\beta}{4}\right) + 1 - \cos\frac{\pi\beta}{4}\right\}$$

and $$\tau = 2\beta/5 .$$

Proof:

Let $E = \{x \mid \frac{1}{2} - \tau \leq |x| \leq \frac{1}{2}\}$

and E' be the complement of E relative to $[-\frac{1}{2}, \frac{1}{2}]$.

Case (i) Suppose

(2.56) $$\int_E f(x)dx \geq \tfrac{1}{2}\alpha$$

(2.57) $$\int_E g(x)dx \geq \tfrac{1}{2}\beta \quad .$$

Then

$$M^* \geq \mathcal{M}^*(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)g(-x)dx$$

$$\geq \int_E f(x)g(-x)dx$$

$$\geq \int_E [f(x) + g(-x) - 1]dx$$

using (2.17).

Hence, $$M^* \geq \tfrac{1}{2}(\alpha + \beta) - m(E)$$

(2.58) $$= \tfrac{1}{2}(\alpha + \beta) - 2\tau \quad .$$

Case (ii) Suppose at least one of (2.56) and (2.57) is not true. Specifically, let us suppose

(2.59) $$\int_E g(x)dx < \tfrac{1}{2}\beta \quad .$$

Then,

$$|F(\pi)| = \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\pi(x + \frac{1}{2})} f(x)dx \right|$$

$$\geq \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin\pi(x + \tfrac{1}{2}) f(x)dx \right|$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos \pi x \, f(x)dx$$

$$\geq 2\int_{\frac{1}{2}-\frac{\alpha}{2}}^{\frac{1}{2}} \cos \pi x dx$$

i.e.,

(2.60) $$|F(\pi)| \geq \frac{2}{\pi}\left(1 - \cos \frac{\pi\alpha}{2}\right) .$$

And similarly,

$$|G(\pi)| \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos \pi x\, g(x) dx$$

$$= 2\int_{\frac{1}{2}-\tau-\frac{\beta}{4}}^{\frac{1}{2}-\tau} \cos \pi x dx + 2\int_{\frac{1}{2}-\frac{\beta}{4}}^{\frac{1}{2}} \cos \pi x dx$$

i.e.,

(2.61) $$|G(\pi)| \geq \frac{2}{\pi}\left(\cos\pi\tau - \cos\pi\left(\tau+\frac{\beta}{4}\right)\right) + \frac{2}{\pi}\left(1 - \cos\frac{\pi\beta}{4}\right)$$

Hence, using (2.47), Lemma 2.7, (2.60) and (2.61),

$$\int_{-1}^{1} |\tilde{\mathcal{M}}(t)| dt \geq |H(\pi)|$$

$$= |F(\pi)|\ |G(\pi)|$$

$$\geq \frac{4}{\pi^2}\left(1-\cos\frac{\pi\alpha}{2}\right)\left\{\cos\pi\tau-\cos\pi\left(\tau+\frac{\beta}{4}\right)+1-\cos\frac{\pi\beta}{4}\right\}$$

Denote the last expression by $2K$, and observe that

$$\int_{-1}^{1} \tilde{\mathcal{M}}(t)dt = 0$$

and
$$\tilde{\mathcal{M}}(t) \geq -\frac{\alpha\beta}{2}, \qquad t \in [-1,1] \quad .$$

Then by Lemma 2.9, for some $t_1 \in [-1,1]$,

$$\tilde{\mathcal{M}}(t_1) \geq \frac{K}{2 - \frac{2K}{\alpha\beta}}$$

i.e.,

$$M^* \geq \frac{\alpha\beta}{2} + \frac{K}{2 - 2K/\alpha\beta} \quad . \tag{2.62}$$

The desired result is now immediate from (2.58), (2.62) and the fact these inequalities are true for all $f \in \mathfrak{F}$, $g \in \mathfrak{G}$.

CHAPTER III

PROBLEM F2

Let $\mathfrak{F}$ denote, as before, the class of all functions f satisfying conditions (2.1) to (2.3). For each $f \in \mathfrak{F}$, define the "complementary" function $g = g_f$ by

$$g(x) = \begin{cases} 1 - f(x), & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases} . \tag{3.1}$$

Define

$$\mathcal{M}(t) = \mathcal{M}(f;t) = \int_0^1 f(x)g(x+t)dt \tag{3.2}$$

$$M = M(f) = \sup_t \mathcal{M}(t) \tag{3.3}$$

$$\lambda_{F2} = \inf_{f \in \mathfrak{F}} M . \tag{3.4}$$

In this chapter we shall find upper and lower estimates for λ_{F2}. We first note that Lemmas 2.1, 2.2 are still valid with $1 - \alpha$ replacing β. That is,

$$\mathcal{M}(t) \text{ is continuous on } (-\infty,\infty) \tag{3.5}$$

$$\int_{-1}^{1} \mathcal{M}(t)dt = \alpha(1 - \alpha) \quad . \tag{3.6}$$

Further, if we define

$$\mathcal{N}(t) = \mathcal{M}(t) + \mathcal{M}(-t) \ , \quad t \geq 0 \tag{3.7}$$

and

$$N = \sup_t \mathcal{N}(t) \tag{3.8}$$

then

$$N \leq 2M \ , \tag{3.9}$$

$$\mathcal{N}(t) \text{ is continuous on } [-1,1] \ , \tag{3.10}$$

$$\int_{0}^{1} \mathcal{N}(t)dt = \alpha(1 - \alpha) \quad . \tag{3.11}$$

We now proceed to obtain lower bounds on λ_{F2} .

<u>LEMMA 3.1</u>

If $0 \leq t \leq 1$,

$$\mathcal{N}(t) \leq 1 - t \quad . \tag{3.12}$$

Proof:

Since
$$f(x) + g(x) = 1 \qquad x \in [0,1]$$
$$f(x+t) + g(x+t) = 1 \quad x \in [0,1-t] \quad ,$$

we have
$$[f(x)+g(x)][f(x+t) + g(x+t)] = 1 \quad .$$

Hence
$$f(x)\,f(x+t) + g(x)g(x+t) \leq 1 \quad .$$

Now,
$$\mathcal{N}(t) = \int_0^{1-t} f(x)g(x+t)dx + \int_t^1 f(x)g(x-t)dx$$
$$= \int_0^{1-t} [f(x)g(x+t) + f(x+t)g(x)]dx$$
$$\leq 1 - t \, .$$

THEOREM 3.1

$$\lambda_{F2} \geq \frac{1 - \sqrt{1 - 2\alpha(1-\alpha)}}{2} \quad . \tag{3.13}$$

<u>Proof</u>:

$$\alpha(1-\alpha) = \int_0^{1-N} \mathcal{N}(t)dt + \int_{1-N}^1 \mathcal{N}(t)dt$$

$$\leq \int_0^{1-N} (1-t)dt + N\int_{1-N}^1 dt$$

by (3.8) and (3.12). This gives

$$\frac{N^2}{2} - N + \alpha(1-\alpha) \leq 0$$

whence

$$N \geq 1 - \sqrt{1 - 2\alpha(1-\alpha)}$$

and hence

$$M \geq \frac{1 - \sqrt{1 - 2\alpha(1-\alpha)}}{2} .$$

Since this is true for all $f \in \mathfrak{F}$, (3.13) follows.

By (6.2), it follows that (3.13) implies the same lower bound for λ_{P2}. This estimate falls short of Świerczkowski's estimate (1.16). However, we will now use what may be called a second moment method to obtain a better estimate than (1.16). We will need the following definitions and lemmas.

Let

$$\bar{f} = \frac{1}{\alpha} \int_0^1 x\, f(x)dx$$

$$\bar{g} = \frac{1}{\beta} \int_0^1 x\, g(x)dx$$

$$\bar{h} = \frac{1}{\alpha\beta} \int_{-1}^{1} t\mathcal{M}(t)dt$$

$$V(f) = \frac{1}{\alpha} \int_0^1 (x - \bar{f})^2 f(x)dx$$

$$V(g) = \frac{1}{\beta} \int_0^1 (x - \bar{g})^2 g(x)dx$$

$$V(h) = \frac{1}{\alpha\beta} \int_{-1}^{1} (t - \bar{h})^2 \mathcal{M}(t)dt$$

where $\beta = 1 - \alpha$.

<u>LEMMA 3.2</u>

(3.14) $$\bar{h} = \bar{g} - \bar{f} \quad .$$

(3.15) $$V(h) = V(f) + V(g) \quad .$$

Proof:

$$\alpha\beta\bar{h} = \int_{-1}^{1} t \int_{o}^{1} f(x)g(x+t)dx$$

$$= \int_{o}^{1} f(x)dx \int_{-1}^{1} t\ g(x+t)dt$$

$$= \int_{o}^{1} f(x)dx \int_{o}^{1} (u-x)g(u)du, \quad (u = x+t)$$

$$= \int_{o}^{1} f(x) \quad (\beta\bar{g} - \beta x)dx$$

$$= \alpha\beta(\bar{g} - \bar{f})$$

which proves (3.14).

$$\alpha\beta V(h) = \int_{-1}^{1} (t-\bar{h})^2 dt \int_{o}^{1} f(x)g(x+t)dx$$

$$= \int_{o}^{1} f(x)dx \int_{-1}^{1} (t-\bar{g}+\bar{f})^2\ g(x+t)dt$$

$$= \int_{o}^{1} f(x)dx \int_{o}^{1} (u-x-\bar{g}+\bar{f})^2\ g(u)du$$

$$= \int_0^1 f(x)\ [\beta V(g) + \beta(u-x)^2]dx$$

$$= \alpha\beta[V(g) + V(f)]$$

which proves (3.15). The change in the order of integration, in both cases, is justified by Fubini's Theorem.

LEMMA 3.3

For all $f \in \mathfrak{F}$,

(3.16) $$\int_0^1 (x - \tfrac{1}{2})^2\, g(x)dx \geq \frac{\beta^3}{12}$$

(3.17) $$V(h) \geq \frac{1}{12}\ \frac{\alpha^2\beta^2}{M^2} \quad .$$

Proof:

The fact that the integral in (3.16) is minimized for the function $g(x) = 1$, $x \in [\frac{1}{2} - \frac{\beta}{2}, \frac{1}{2} + \frac{\beta}{2}]$ and $= 0$ elsewhere; and similarly $V(h)$ is minimized when $\mathcal{M}(t) = 1$ for $t \in [\bar{h} - \frac{\alpha\beta}{2M}, \bar{h} + \frac{\alpha\beta}{2M}]$ and $= 0$ elsewhere, provides a heuristic argument for these results. However, a proof of (3.16) will be presented here while that of (3.17), being the same except for notational adjustments, is omitted.

Let $A = [\frac{1}{2} - \frac{\beta}{2}, \frac{1}{2} + \frac{\beta}{2}]$ and A' be the complement of A relative to $[0,1]$. Then

$$\max_{x \in A} (x - \tfrac{1}{2})^2 = \frac{\beta^2}{4}$$

$$\min_{x \in A'} (x - \tfrac{1}{2})^2 = \frac{\beta^2}{4} .$$

Hence, by a mean-value theorem for integrals,

$$\int_0^1 (x - \tfrac{1}{2})^2 g(x)dx = \int_A (x - \tfrac{1}{2})^2 [g(x)-1]dx$$

$$+ \int_{A'} (x - \tfrac{1}{2})^2 g(x)dx + \int_A (x - \tfrac{1}{2})^2 dx$$

$$\geq \frac{\beta^2}{4} \int_A [g(x)-1]dx + \frac{\beta^2}{4} \int_{A'} g(x)dx + \frac{\beta^3}{12}$$

$$= \frac{\beta^3}{12} + \frac{\beta^2}{4} \left\{ \int_{A \cup A'} g(x)dx - \int_A dx \right\}$$

$$= \frac{\beta^3}{12}$$

since the quantity in the braces is zero.

THEOREM 3.2

Suppose, without loss of generality, that $\alpha \leq \frac{1}{2}$. Then

$$\lambda_{F2} \geq \frac{\alpha(1-\alpha)}{\sqrt{4-5\alpha+2\alpha^2}} \quad . \tag{3.17}$$

Proof:

$$\frac{\alpha^3\beta^3}{12M^2} \leq \alpha\beta\, V(h)$$

$$= \alpha\beta\{V(f) + V(g)\}$$

$$= \beta\int_0^1 (x-\bar{f})^2 f(x)dx + \alpha\int_0^1 (x-\bar{g})^2 g(x)dx$$

$$\leq \beta\int_0^1 (x-\tfrac{1}{2})^2 f(x)dx + \alpha\int_0^1 (x-\tfrac{1}{2})^2 g(x)dx \quad .$$

Here, we have used the well-known and easily proved fact that the second moment is minimal when taken about the mean. Therefore,

$$\frac{\alpha^3\beta^3}{12M^2} \leq \beta\int_0^1 (x-\tfrac{1}{2})^2 f(x)dx + \alpha\int_0^1 (x-\tfrac{1}{2})^2 g(x)dx$$

$$= \beta\int_0^1 (x-\tfrac{1}{2})^2[f(x)+g(x)]dx - (\beta-\alpha)\int_0^1 (x-\tfrac{1}{2})^2 g(x)dx$$

Since $f(x) + g(x) = 1$, and by (3.16)

$$\frac{\alpha^3\beta^3}{12M^2} \leq \frac{\beta}{12} - (\beta - \alpha)\frac{\beta^3}{12} .$$

On writing $\beta = 1 - \alpha$, this gives

$$M \geq \frac{\alpha(1-\alpha)}{\sqrt{4 - 5\alpha + 2\alpha^2}}$$

and hence the required result.

From (6.2) and (3.17), it follows that

$$(3.18) \qquad \lambda_{P2} \geq \frac{\alpha(1-\alpha)}{\sqrt{4 - 5\alpha + 2\alpha^2}} .$$

We now wish to prove that (3.18) is sharper than Świerczkowski's result (1.16). Let us denote the right-hand expressions in (1.16) and (3.18) by $E_1(\alpha)$ and $E_2(\alpha)$ respectively. Then the power-series developments are

$$E_1(\alpha) = \frac{\alpha}{2} - \frac{3}{16}\alpha^2 - \frac{15}{64}\alpha^3 - \ldots.$$

$$E_2(\alpha) = \frac{\alpha}{2} - \frac{3}{16}\alpha^2 - \frac{37}{256}\alpha^3 - \ldots.$$

which shows that $E_2(\alpha) > E_1(\alpha)$ for sufficiently small values of α. But even more is true:

(3.19) $$E_2(\alpha) > E_1(\alpha) \quad \text{for all} \quad 0 < \alpha \le \tfrac{1}{2} \ .$$

For, (3.19) holds if and only if

$$25\alpha^2(1-\alpha)^2 > 4(4-5\alpha+2\alpha^2)(1 - \sqrt{1-5\alpha(1-\alpha)/2})^2$$

iff $$25\alpha^2(1-\alpha)^2 > 4(4-5\alpha+2\alpha^2)\{2 - \tfrac{5}{2}\alpha(1-\alpha)-2\sqrt{1-5\alpha(1-\alpha)/2}\}$$

iff $$8(4-5\alpha+2\alpha^2)\sqrt{1-5\alpha(1-\alpha)/2} > 2(4-5\alpha+2\alpha^2)\{4-5\alpha(1-\alpha)\}-25\alpha^2(1-\alpha)^2$$

iff $$64(4-5\alpha+2\alpha^2)^2\left(1 - \frac{5\alpha(1-\alpha)}{2}\right) > 4(4-5\alpha+2\alpha^2)^2\{4-5\alpha(1-\alpha)\}^2$$
$$+ 625\alpha^4(1-\alpha)^4-100\alpha^2(1-\alpha)^2(4-5\alpha+2\alpha^2)\{4-5\alpha(1-\alpha)\}$$

iff $$-4(4-5\alpha+2\alpha^2)^2 > 25\alpha^2(1-\alpha)^2-4(4-5\alpha+2\alpha^2)\{4-5\alpha(1-\alpha)\}$$

iff $$12(4-5\alpha+2\alpha^2) > 25(1-\alpha)^2$$

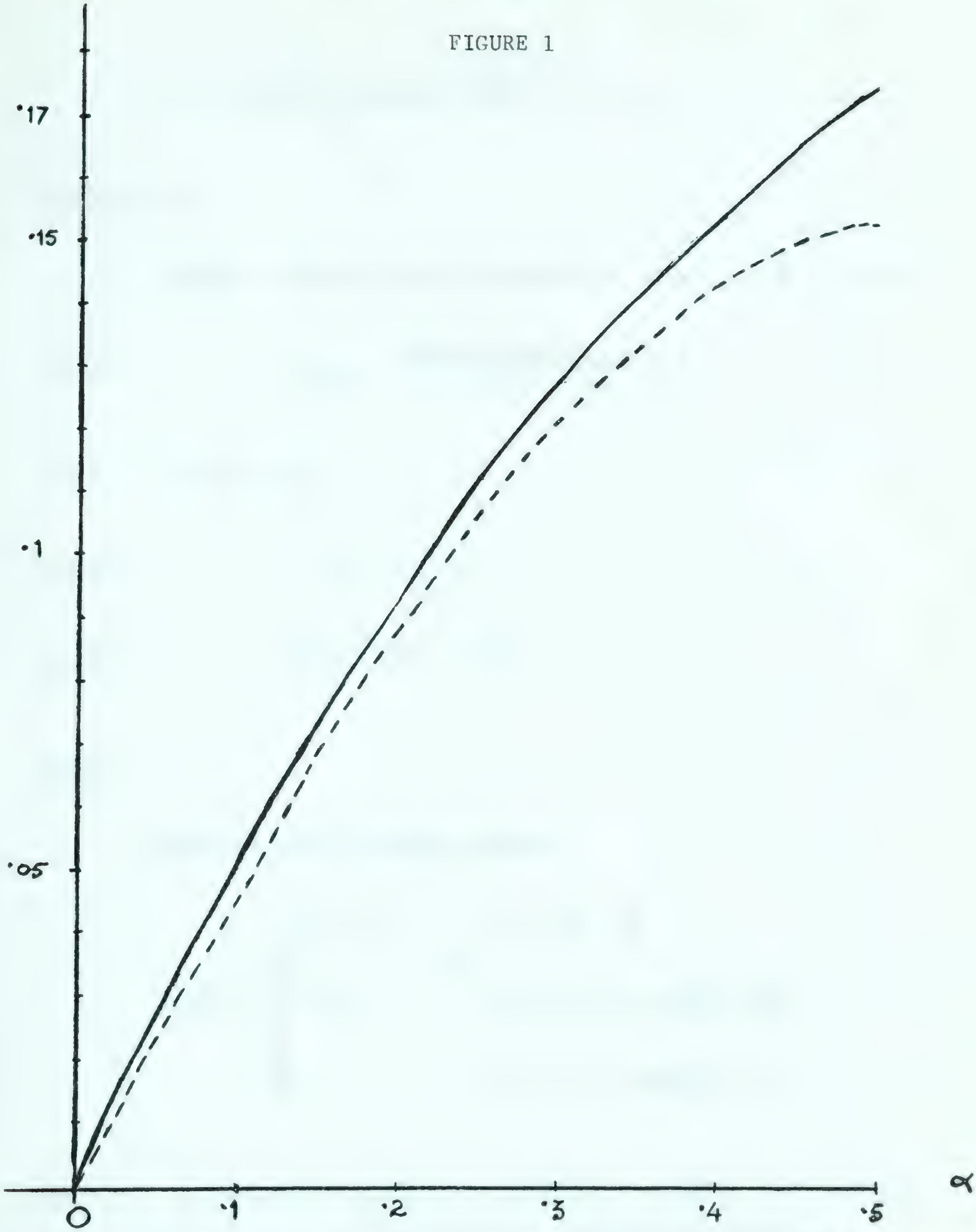
iff $$23 - 10\alpha - \alpha^2 > 0$$

which is true for all values of α in question. This proves (3.19), and as a consequence, that (3.18) is an improvement over (1.16).

We may take note of the fact that from (3.18) one can deduce (see (6.4) in Appendix)

(3.20) $$\lambda_{N2} \ge \frac{k(n-k)}{\sqrt{4n^2-5nk+2k^2}} \ .$$

FIGURE 1



GRAPHICAL COMPARISON BETWEEN ESTIMATES $E_1(\alpha)$ and $E_2(\alpha)$.

- - - - - $E_1(\alpha)$

——— $E_2(\alpha)$

Now to obtain an upper bound for λ_{F2} .

THEOREM 3.3

Suppose, without loss of generality, that $\alpha \le \frac{1}{2}$. Then

$$\lambda_{F2} \le \frac{2(2\alpha-a)(1-a-\alpha)+\alpha+a}{6} \tag{3.21}$$

where a is given by

$$0 \le a \le \alpha \tag{3.22}$$

$$6a\alpha - 3a^2 = 4\alpha^2 - \alpha \ . \tag{3.23}$$

Proof:

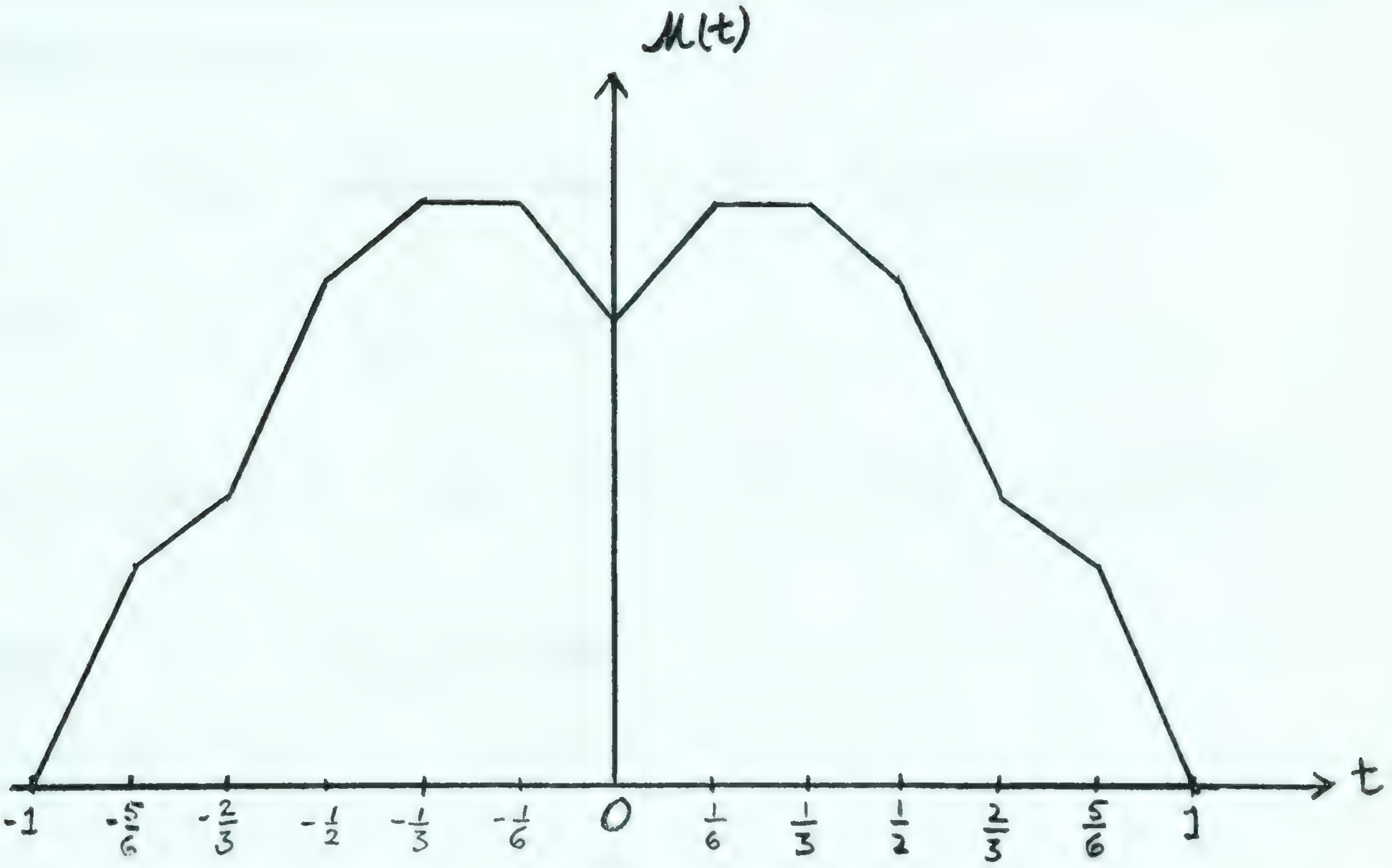
Consider the following function

$$f(x) = \begin{cases} 2\alpha - a , & x \in [\frac{1}{3} , \frac{2}{3}] \\ \alpha , & x \in (\frac{1}{6} , \frac{1}{3}) \cup (\frac{2}{3} , \frac{5}{6}) \\ a , & x \in [0 , \frac{1}{6}] \cup [\frac{5}{6} , 1] \end{cases}$$

where a is defined by (3.21) and (3.22). The $\mathcal{M}(t)$ for this f is a symmetric broken-line function as illustrated in Figure 2. It is easily verified that

$$\mathcal{M}(0) \le \mathcal{M}(\tfrac{1}{6}) = \mathcal{M}(\tfrac{1}{3}) \ge \mathcal{M}(\tfrac{1}{2}) \ge \mathcal{M}(\tfrac{2}{3}) \ge \mathcal{M}(\tfrac{5}{6})$$

FIGURE 2



$\mu(t)$
t
-1
$-\frac{5}{6}$
$-\frac{2}{3}$
$-\frac{1}{2}$
$-\frac{1}{3}$
$-\frac{1}{6}$
0
$\frac{1}{6}$
$\frac{1}{3}$
$\frac{1}{2}$
$\frac{2}{3}$
$\frac{5}{6}$
1

and hence $\mathcal{M}(t)$ takes its maximum at $t = \frac{1}{3}$. Thus,

$$M = \mathcal{M}(\tfrac{1}{3}) = \frac{2(2\alpha-a)(1-\alpha-a) + \alpha + a}{6}.$$

This proves (3.20).

It must be pointed out that (3.20) is useful only for values of α in a "small" neighbourhood of $\frac{1}{2}$. For other values, it gives a rather poor bound.

For the particular case $\alpha = \frac{1}{2}$, (3.20) gives

(3.24) $$\lambda_{F2} \leq \cdot 1944 .$$

For this case, W. O. J. Moser [5] has found a function to show that

(3.25) $$\lambda_{F2} \leq \cdot 1933 .$$

CHAPTER IV

TWO-VARIABLE CASE OF F1 AND F2

In this chapter, we consider the two-dimensional analogues of F1 and F2.

Let $\mathcal{F}$ denote the class of all functions $f(x,y)$ such that

$$f(x,y) \in L[0,1;0,1] \tag{4.1}$$

$$\begin{cases} 0 \le f(x,y) \le 1 & \text{when } (x,y) \in [0,1;0,1] \\ f(x,y) = 0 & \text{otherwise} \end{cases} \tag{4.2}$$

$$\int_0^1\int f(x,y)\,dxdy = \alpha \quad , \tag{4.3}$$

and $\mathcal{G}$ be the class of all functions $g(x,y)$ such that

$$g(x,y) \in L[0,1;0,1] \tag{4.4}$$

$$\begin{cases} 0 \le g(x,y) \le 1 & \text{when } (x,y) \in [0,1;0,1] \\ g(x,y) = 0 & \text{otherwise} \end{cases} \tag{4.5}$$

$$\int_0^1\int g(x,y)\,dxdy = \beta \quad . \tag{4.6}$$

Let

$$(4.7) \qquad \mathcal{M}(t_1,t_2) = \int_0^1\!\!\int f(x,y)g(x+t_1,y+t_2)dxdy$$

$$(4.8) \qquad M = \sup_{t_1,t_2} \mathcal{M}(t_1,t_2)$$

$$(4.9) \qquad \lambda_{F1}^{(2)} = \inf_{\substack{f\in\mathcal{F}\\ g\in\mathcal{G}}} M \quad .$$

We can now prove

(4.10) $\mathcal{M}(t_1,t_2)$ is a continuous function of both variables,

$$(4.11) \qquad \int_1^1\!\!\int \mathcal{M}(t_1,t_2)dt_1dt_2 = \alpha\beta \quad ,$$

and

$$(4.12) \qquad \mathcal{M}(t_1,t_2) \le (1-|t_1|)(1-|t_2|) \text{ if } (t_1,t_2)\in[-1,1;-1,1] \quad .$$

Since

$$\mathcal{M}(t_1+h,t_2+k) - \mathcal{M}(t_1,t_2)$$

$$= \int_0^1\!\!\int f(x,y)[g(x+t_1+h,y+t_2+k)-g(x+t_1,y+t_2)]dxdy \ ,$$

$$|\mathcal{M}(t_1+h,t_2+k) - \mathcal{M}(t_1,t_2)|$$

$$\le \int_0^1\!\!\int |g(x+t_1+h,y+t_2+k)-g(x+t_1,y+t_2)|dxdy \quad .$$

By the mean-continuity property, the last integral tends to zero, as $h \to 0$, $k \to 0$. This proves (4.10). Now,

$$\int_{-1}^{1}\int \mathcal{M}(t_1, t_2)\,dt_1 dt_2$$

$$= \int_{-1}^{1}\int dt_1 dt_2 \int_{0}^{1}\int f(x,y)g(x+t_1, y+t_2)\,dxdy$$

$$= \int_{0}^{1}\int f(x,y)\,dxdy \int_{-1}^{1}\int g(x+t_1, y+t_2)\,dt_1 dt_2$$

$$= \int_{0}^{1}\int f(x,y)\,dxdy \int_{0}^{1}\int g(u,v)\,dudv$$

$$= \alpha\beta$$

which proves (4.11).

(Fubini's theorem justifies the change of order of integration).

Finally, let $t_1 \geq 0$, $t_2 \geq 0$. Then

$$\mathcal{M}(t_1, t_2) = \int_{0}^{1-t_1} \int_{0}^{1-t_2} f(x,y)g(x+t_1, y+t_2)\,dxdy$$

$$\leq (1-t_1)(1-t_2)$$

since the integrand is less than 1. Similarly when one or both t_1, t_2 are negative. (4.12) is thus proved.

From (4.11) it follows immediately that

$$\lambda_{F1}^{(2)} \geq \frac{\alpha\beta}{4} \tag{4.13}$$

which corresponds to (2.12). However, we can do better by making use of (4.12).

THEOREM 4.1

$$\lambda_{F1}^{(2)} \geq 1 - \sqrt[4]{1 - \alpha\beta} \quad . \tag{4.14}$$

Proof:

Let

$$Q = [-1,1;-1,1]$$

$$R = [-1+M,1-M;-1+M,1-M]$$

Then,

$$\alpha\beta = \iint_R \mathcal{M}(t_1,t_2)dt_1dt_2 + \iint_{Q-R} \mathcal{M}(t_1,t_2)dt_1dt_2$$

$$\leq M\iint_R dt_1dt_2 + \iint_{Q-R} (1-|t_1|)(1-|t_2|)dt_1dt_2$$

$$= 4M(1-M)^2 + \iint_Q (1-|t_1|)(1-|t_2|)dt_1dt_2$$

$$- \iint_R (1-|t_1|)(1-|t_2|)dt_1dt_2$$

$$= 4M(1-M)^2 + 1 - (1-M^2)^2$$

$$= 1 - (1-M)^4 \quad .$$

This gives

$$(1 - M)^4 \geq 1 - \alpha\beta$$

i.e.,

$$M \geq 1 - \sqrt[4]{1 - \alpha\beta}$$

from which (4.14) follows.

To find an upper bound for $\lambda^{(2)}_{F1}$, suppose that $\alpha \leq \beta$. Consider the function $f(x,y)$ which takes value 1 on the set which is the cartesian product of (the interval—sum) $[0, \frac{\sqrt{\alpha}}{2}] \cup [1 - \frac{\sqrt{\alpha}}{2}, 1]$ with itself and takes the value zero elsewhere. Let $g(x,y)$ be the function which equals $\beta/(1 - \frac{\sqrt{\alpha}}{2})^2$ on the square $\left[\frac{\sqrt{\alpha}}{4}, 1 - \frac{\sqrt{\alpha}}{4} ; \frac{\sqrt{\alpha}}{4}, 1 - \frac{\sqrt{\alpha}}{4}\right]$ and 0 elsewhere. This pair of functions shows that

$$(4.15) \qquad \lambda^{(2)}_{F1} \leq \frac{\alpha\beta}{(2 - \sqrt{\alpha \wedge \beta})^2}$$

Consider now the analogue of F2: Let $\mathfrak{F}$ denote the class of all f satisfying (4.1), (4.2) and (4.3), and for each $f \in \mathfrak{F}$ define $g = g_f$ by

$$(4.16)\quad g(x,y) = \begin{cases} 1 - f(x,y) & (x,y) \in [0,1;0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Further, let

$$(4.17)\quad \mathcal{M}(t_1,t_2) = \mathcal{M}(f;t_1,t_2) = \int_0^1\!\!\int f(x,y)g(x+t_1,y+t_2)dxdy$$

$$(4.18)\quad M = M(f) = \sup_{t_1,t_2} \mathcal{M}(t_1,t_2)$$

$$(4.19)\quad \lambda_{F2}^{(2)} = \inf_{f\in\mathfrak{F}} M .$$

If also we define

$$\mathcal{N}(t_1,t_2) = \mathcal{M}(t_1,t_2) + \mathcal{M}(-t_1,t_2) + \mathcal{M}(t_1,-t_2) + \mathcal{M}(-t_1,-t_2)$$

$$\text{for } t_1 \geq 0, \quad t_2 \geq 0,$$

$$N = \sup_{t_1,t_2} \mathcal{N}(t_1,t_2) ,$$

then

$$N \leq 4M .$$

By imitating the proofs of (4.10) and (4.11) and keeping in mind $\beta = 1 - \alpha$, one can prove

(4.20) $\mathcal{N}(t_1,t_2)$ is a continuous function of both variables,

(4.21) $$\int_0^1\int \mathcal{N}(t_1,t_2)dt_1dt_2 = \alpha(1-\alpha) \quad .$$

We will further prove that

(4.22) $$\mathcal{N}(t_1,t_2) \leq (1-t_1)(1-t_2), \quad 0 \leq t_1 \leq 1, \quad 0 \leq t_2 \leq 1 \quad .$$

For, from the equations

$$f(x,y) + g(x,y) = 1, \quad (x,y) \in [0,1;0,1]$$

$$f(x+t_1,y)+g(x+t_1,y) = 1, \quad (x,y) \in [0,1-t_1;0,1]$$

$$f(x,y+t_2)+g(x,y+t_2) = 1, \quad (x,y) \in [0,1;0,1-t_2]$$

$$f(x+t_1,y+t_2)+g(x+t_1,g+t_2) = 1, \quad (x,y) \in [0,1-t_1;0,1-t_2]$$

we obtain

(4.23) $$f(x,y)g(x+t_1,y+t_2)+f(x+t_1,y)g(x,y+t_2)+f(x,y+t_2)g(x+t_1,y)$$
$$+f(x+t_1,y+t_2)g(x,y) \leq 1 \quad .$$

Hence,

$$\mathcal{N}(t_1,t_2) = \int_0^{1-t_1}\int_0^{1-t_2} f(x,y)g(x+t_1,y+t_2)dxdy$$

$$+ \int_0^{1-t_1}\int_0^{1-t_2} f(x+t_1,y)g(x,y+t_2)dxdy$$

$$+ \int_0^{1-t_1}\int_0^{1-t_2} f(x,y+t_2)g(x+t_1,y)dxdy$$

$$+ \int_0^{1-t_1}\int_0^{1-t_2} f(x+t_1,y+t_2)g(x,y)dxdy$$

$$\leq (1-t_1)(1-t_2)$$

by (4.23), thus proving (4.22).

THEOREM 4.2

$$\lambda_{F2}^{(2)} \geq \frac{1 - \sqrt[4]{1 - 4\alpha(1-\alpha)}}{4} \quad . \tag{4.24}$$

Proof: Let

$$Q' = [0,1;0,1]$$

$$R' = [0,1-N;0,1-N] \quad .$$

Then,

$$\alpha(1-\alpha) = \iint_{R'} \mathcal{N}(t_1,t_2)dt_1dt_2 + \iint_{Q'-R'} \mathcal{N}(t_1,t_2)dt_1dt_2$$

$$\leq N\iint_{R'} dt_1dt_2 + \iint_{Q'-R'} (1-t_1)(1-t_2)dt_1dt_2$$

$$= N(1-N)^2 + \iint_{Q'} (1-t_1)(1-t_2)dt_1dt_2 - \iint_{R'}(1-t_1)(1-t_2)dt_1dt_2$$

$$= N(1-N)^2 + \frac{1}{4} - \frac{(1-N^2)^2}{4}$$

$$= \frac{1-(1-N)^4}{4}$$

which implies

$$N \geq 1 - \sqrt[4]{1-4\alpha(1-\alpha)}$$

and hence

$$M \geq \frac{1-\sqrt[4]{1-4\alpha(1-\alpha)}}{4}$$

from which (4.24) follows immediately.

It is possible to obtain improvements over (4.14) and (4.24) by changing the region of integration, although the explicit form of the expressions in (4.14) and (4.24) is then lost. Specifically, we prove that

THEOREM 4.3

(4.25) $$\lambda_{F1}^{(2)} \geq Z_1$$

where Z_1 is that zero of

$$x^2(3 - 2\log x) - 4x + \alpha\beta$$

which lies between 0 and 1.

(4.26) $$\lambda_{F2}^{(2)} \geq Z_2/4$$

where Z_2 is that zero of

$$x^2(3 - 2\log x) - 4x + 4\alpha(1-\alpha)$$

which lies between 0 and 1.

Proof: Let S be the region consisting of all points (t_1,t_2) such that

$$|t_1| \leq 1$$

$$|t_2| \leq 1$$

$$(1 - |t_1|)(1 - |t_2|) > M \ .$$

In the proof of Theorem 4.1, replacing R by S, we obtain

$$\alpha\beta \leq M \iint_S dt_1 dt_2 + \iint_Q (1-|t_1|)(1-|t_1|)dt_1 dt_2$$

$$- \iint_S (1-|t_1|)(1-|t_2|)dt_1 dt_2$$

$$= 4M(1-M+M\log M) + = - (1-M^2+2M^2\log M)$$

$$= 4M - M^2(3-2\log M)$$

i.e.,

(4.27) $$M^2(3-2\log M) - 4M + \alpha\beta \leq 0 \quad .$$

Now let

$$\Phi(M) = M^2(3-2\log M) - 4M + \alpha\beta, \quad 0 < M \leq 1$$

$$\Phi(0) = \alpha\beta \quad .$$

Considering the derived functions $\Phi'(M)$, $\Phi''(M)$, it is easy to see that

$$\Phi''(M) > 0 \quad , \qquad 0 < M \leq 1$$

and hence that $\Phi'(M)$ is a strictly increasing function in the interval $[0,1]$. Since $\Phi'(1) = 0$, $\Phi'(M)$ must be < 0 in the interval $(0,1)$ and hence $\Phi(M)$ is a strictly decreasing function in the interval $(0,1)$. It now follows that $\Phi(M)$ has exactly one

zero, say Z_1, in the interval $(0,1)$ and that in order for (4.27) to hold we must have

$$M \geq Z_1 \quad .$$

This proves (4.25). Similarly, one can prove (4.26) by replacing R' in the proof of Theorem 4.2 by the region S' consisting of points (t_1,t_2) such that

$$t_1 \geq 0$$

$$t_2 \geq 0$$

$$(1-t_1)(1-t_2) > N \quad .$$

The details are omitted.

CHAPTER V

PROBLEM OF CZIPSZER

In this chapter, we deal with the generalizations of Czipszer's problem, first to functions of one variable and later to functions of several variables.

Let Φ be the class of all functions φ such that

$$\varphi \in L(-\infty,\infty) \tag{5.1}$$

$$0 \leq \varphi(x) \leq 1 \tag{5.2}$$

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1 \ . \tag{5.3}$$

Let

$$\mathcal{M}(t) = \mathcal{M}(\varphi;t) = \int_{-\infty}^{\infty} \varphi(x)\varphi(x+t)dx \tag{5.4}$$

$$m = m(\varphi) = \inf_{|t|\leq 1} \mathcal{M}(t) \tag{5.5}$$

and

$$\mu_F = \inf_{\varphi\in\Phi} \{1-m\} \ . \tag{5.6}$$

The problem is to find or estimate μ_F.

We can now prove:

(5.7) $$0 \leq \mathcal{M}(t) \leq 1 ,$$

(5.8) $\mathcal{M}(t)$ is an even function: $\mathcal{M}(t) = \mathcal{M}(-t)$,

(5.9) $\mathcal{M}(t)$ is continuous everywhere,

and

(5.10) $$\int_{-\infty}^{\infty} \mathcal{M}(t)dt = 1 .$$

(5.7) and (5.8) are obvious, while the proofs of (5.9) and (5.10) are the same as proofs of Lemmas 2.1 and 2.2, except for minor changes.

By (5.7) and (5.10),

$$\int_{-1}^{1} \mathcal{M}(t)dt \leq 1$$

and hence

$$m \leq \frac{1}{2}$$

for all $\varphi \in \Phi$. Therefore,

(5.11) $$\mu_F \geq \frac{1}{2}$$

which corresponds to (1.22).

THEOREM 5.1

$$0{\cdot}5892 \leq \mu_F \leq \frac{2}{3} \ . \tag{5.12}$$

Proof: Consider the function

$$\psi(x) = \begin{cases} 1 & x \in [0,\frac{2}{3}] \cup [1,\frac{4}{3}] \\ 0 & \text{elsewhere} \end{cases}$$

For this function $m = \frac{1}{3}$, which proves that $\mu_F \leq \frac{2}{3}$. (This example was communicated to us by Świerczkowski.)

The following method has been adapted from the one used by A. Rényi [7].

From the easily verified identity

$$\int_{-\infty}^{\infty} \mathcal{M}(t) \cos \lambda t dt = \left(\int_{-\infty}^{\infty} \varphi(x) \cos \lambda x dx \right)^2 + \left(\int_{-\infty}^{\infty} \varphi(x) \sin \lambda x dx \right)^2$$

it follows that, for all real λ,

$$\int_{-\infty}^{\infty} \mathcal{M}(t) \cos \lambda t dt \geq 0 \ . \tag{5.13}$$

Define

$$\delta(t) = \begin{cases} \mathcal{M}(t) - m, & t \in [-1,1] \\ \mathcal{M}(t) & \text{otherwise} \ . \end{cases} \tag{5.14}$$

Then

$$\delta(t) \geq 0 \quad \text{for all} \quad t, \tag{5.15}$$

and

$$\int_{-\infty}^{\infty} \mathcal{M}(t)\cos \lambda t dt = m\int_{-1}^{1} \cos \lambda t dt + \int_{-\infty}^{\infty} \delta(t)\cos \lambda t dt \ . \tag{5.16}$$

Writing $\lambda = 0$ in (5.16),

$$\int_{-\infty}^{\infty} \delta(t)dt = 1-2m \ . \tag{5.17}$$

Writing $\lambda = \lambda_o = \pi + 1\cdot352$ in the same equation, and using (5.13),

$$0 \leq m \cdot \frac{2\sin \lambda_o}{\lambda_o} + \int_{-\infty}^{\infty} \delta(t)\cos \lambda_o t \ dt$$

$$\leq m \cdot \frac{2\sin \lambda_o}{\lambda_o} + \int_{-\infty}^{\infty} \delta(t)dt$$

$$\leq -(0\cdot2172)m + 1 - 2m$$

from which

$$m \leq 0\cdot4107$$

i.e.,

$$1 - m \geq 0\cdot5892 \ .$$

Since this is true for all $\varphi \in \Phi$,

$$\mu_F \geq 0 \cdot 5892 \ .$$

Both the problem and the method lend themselves to direct generalization to k variables. For this purpose, it is convenient to use the vector notation. Let E_k denote the k-dimensional real Euclidean space. We shall use the notation

$$\bar{x} = (x_1, x_2, \ldots, x_k)$$

to denote the elements of E_k .

Let Φ denote the class of all real-valued functions φ defined on E_k and satisfying the following conditions:

$$\varphi(\bar{x}) \in L(E_k) \tag{5.18}$$

$$0 \leq \varphi(\bar{x}) \leq 1 \tag{5.19}$$

$$\int_{E_k} \varphi(\bar{x})\, d\bar{x} = 1. \tag{5.20}$$

Define

$$\mathcal{M}(\bar{t}) = \mathcal{M}(\varphi;\bar{t}) = \int_{E_k} \varphi(\bar{x})\varphi(\bar{x}+\bar{t})d\bar{x} \ , \tag{5.21}$$

$$m = m(\varphi) = \inf_{\bar{t}\in K} \mathcal{M}(\bar{t}) \tag{5.22}$$

where K denotes the hypercube $|t_i| \leq 1$, $i = 1,2,\ldots,k$,

$$\mu_F^{(k)} = \inf_{\varphi\in\Phi} (1-m) \quad . \tag{5.23}$$

The proofs of (5.9) and (5.10), with minor changes will show that

(5.24) $\mathcal{M}(\bar{t})$ is a (non-negative) continuous function of $\bar{t}$,

$$\int_{E_k} \mathcal{M}(\bar{t})d\bar{t} = 1 \quad . \tag{5.25}$$

From (5.25), one obtains

$$m \leq \frac{1}{2^k}$$

and hence

$$\mu_F^{(k)} \geq 1 - \frac{1}{2^k} \tag{5.26}$$

of which (5.11) is a special case. Finally, we prove

<u>THEOREM 5.2</u>

$$1 - \frac{0\cdot8215}{2^k} \leq \mu_F^{(k)} \leq 1 - \frac{1}{3^k} \quad . \tag{5.27}$$

<u>Proof</u>: With the usual dot-product notation, viz.,

$$\bar{\lambda} \cdot \bar{t} = \lambda_k t_1 + \dots + \lambda_k t_k ,$$

we assert that, for all $\bar{\lambda} \in E_k$,

$$\int_{E_k} \mathcal{M}(\bar{t}) \cos(\bar{\lambda}\cdot\bar{t})d\bar{t} \geq 0 . \tag{5.28}$$

(This follows from the fact that the left-hand side equals $\left| \int_{E_k} \varphi(\bar{x})\, e^{i(\bar{\lambda}\cdot\bar{x})}\, d\bar{x}\right|^2$.)

If we define

$$\delta(\bar{t}) = \begin{cases} \mathcal{M}(\bar{t})-m, & \bar{t} \in K \\ \mathcal{M}(\bar{t}) & \text{otherwise} \end{cases} \tag{5.29}$$

then

$$\delta(\bar{t}) \geq 0 \quad \text{for all } \bar{t} \in E_k \tag{5.30}$$

and

$$\int_{E_k} \mathcal{M}(\bar{t})\cos(\bar{\lambda}\cdot\bar{t})d\bar{t} = m\int_K \cos(\bar{\lambda}\cdot\bar{t})d\bar{t} + \int_{E_k} \delta(\bar{t})\cos(\bar{\lambda}\cdot\bar{t})d\bar{t} . \tag{5.31}$$

Substituting $\bar{\lambda} = 0$ in (5.31) gives

$$\int_{E_k} \delta(\bar{t})\, d\bar{t} = 1 - 2^k . m ,$$

while putting $\lambda_1 = \pi + 1\cdot352$, $\lambda_2 = \lambda_3 = \ldots \lambda_k = 0$ in the same equation and using (5.28),

$$0 \leq - 2^k m \frac{\sin(1\cdot352)}{\pi + 1\cdot352} + 1 - 2^k m$$

i.e.,

$$m \leq \frac{0\cdot8215}{2^k}$$

and hence

$$\mu_F^{(k)} \geq 1 - \frac{0\cdot8215}{2^k} .$$

The second part is proved by considering the function which takes the value 1 in the region R and 0 elsewhere, where R is the cartesian product of the interval-sum $[0, \frac{2}{3}] \cup [1, \frac{4}{3}]$ with itself k times.

APPENDIX

In this section, we shall compare λ_{N1}, λ_{P1} and λ_{F1} ; and similarly for other problems.

Problem P1 can also be stated in terms of characteristic functions. That is, in the statement of Problem P1, we may replace the sets X and Y by their characteristic functions, the measure by the integral and $m(X \cap Y_t)$ by $\int_0^1 f(x)g(x-t)dx$ (where f,g are characteristic functions of X,Y respectively). Since the classes of these characteristic functions are subclasses of $\mathcal{F}$ and $\mathcal{G}$ respectively, it follows that

$$\lambda_{P1} \geq \lambda_{F1} \quad . \tag{6.1}$$

Similarly,

$$\lambda_{P2} \geq \lambda_{F2} \tag{6.2}$$

$$\mu_P \geq \mu_F \quad . \tag{6.3}$$

Consequently, any lower bound for λ_{F1}, λ_{F2}, μ_F implies the same lower bound for λ_{P1}, λ_{P2}, μ_P respectively.

In [9], Świerczkowski has proved, in effect, that a lower bound on λ_{N2} implies a "similar" bound on λ_{P2}. More precisely,

$$\lambda_{N2} \geq F(\tfrac{k}{n}, \tfrac{\ell}{n})$$

$$\Longrightarrow \qquad \lambda_{P2} \geq \frac{F(\alpha,\beta)}{n}$$

($\ell = n-k$; $\beta = 1-\alpha$).

The method, and hence the conclusion, is still valid if we replace N2 and P2 by N1 and P1 respectively.

Conversely, we can prove that

$$\lambda_{P1} \geq F(\alpha,\beta)$$

$$\Longrightarrow \qquad \lambda_{N1} \geq n\,F(\tfrac{k}{n}, \tfrac{\ell}{n})$$

and a similar statement for $\lambda_{P2}, \lambda_{N2}$.

For let $A = \{a_i\}_1^k$, $B = \{b_j\}_1^\ell$ be any subsets of $\{1,2,\ldots,n\}$. Define

$$X = \bigcup\{[\tfrac{i-1}{n}, \tfrac{i}{n}] \mid i \in A\}$$

$$Y = \bigcup\{[\tfrac{i-1}{n}, \tfrac{i}{n}] \mid i \in B\} \quad .$$

Then, $m(X) = k/n$, $m(Y) = \ell/n$; and the hypothesis implies that there exists a real $0 \leq t_o \leq 1$ such that

$$m(X \cap Y_{t_o}) \geq F(\frac{k}{n}, \frac{\ell}{n}) .$$

Now, $m(X \cap Y_t)$ as a function of t is a broken-line function with corners only at the points of the form $\frac{i}{n}$, i an integer. Therefore, we may assume that $t_o = j/n$ for some integer j, $0 \leq j \leq n$. Thus,

$$m(X \cap Y_{j/n}) \geq F(\frac{k}{n}, \frac{\ell}{n}) .$$

Observing that

$$|A \cap B_j| = n \cdot m(X \cap Y_{j/n})$$

we obtain

$$|A \cap B_j| \geq n \cdot F(\frac{k}{n}, \frac{\ell}{n})$$

from which the required conclusion now follows. The argument also applies in the special case when A and B are complementary sets.

We can now combine the two statements above into one:

For $i = 1,2$,

(6.4) $$\lambda_{Pi} \geq F(\alpha,\beta) \Longleftrightarrow \frac{\lambda_{Ni}}{n} \geq F(\frac{k}{n}, \frac{\ell}{n}) .$$

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